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CA DEPT OF STATISTICS A E GELFAND ET AL. 06 APR 88
TR-403 N00014-86-K-0156

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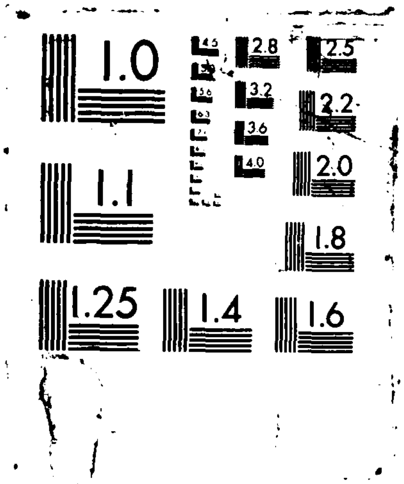
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ON THE ESTIMATION OF A VARIANCE RATIO

BY

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TECHNICAL REPORT NO. 403
APRIL 6, 1988

Prepared Under Contract
N00014-86-K-0156 (NR-042-267)
For the Office of Naval Research

Herbert Solomon, Project Director

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

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Key Words and Phrases: variance ratio; loss function; invariance; admissibility; inadmissibility.

ABSTRACT

The estimation of the ratio of two independent normal variances is considered under scale invariant squared error loss function, when the means are unknown. The best invariant estimator is shown to be inadmissible. Two new classes of improved estimators are obtained, one by extending Stein (1964) and the other by extending Brown (1968). Numerical studies are presented to indicate the percent improvements in risk.

1. INTRODUCTION

Let X_{ij}, $j = 1,\ldots,n_i$, and $n_i \geq 6$ be random samples from independent normal distributions with mean ξ_i and variance σ_i^2, $i = 1,2$. We consider the problem of estimation of the variance ratio $\theta = \sigma_1^2/\sigma_2^2$. In fact, our discussion allows immediate extension to more general parametric functions like $\sigma_1^{m_1}\sigma_2^{m_2}$, where m_1 and m_2 are arbitrary.

This problem is motivated by the work of Stein (1964) and Brown (1968). Stein (1964) proved that for a single sample of X_i's, the usual estimator is inadmissible for estimating σ^2 under

the squared error loss by proving an estimator which has smaller risk (expected loss). Brown considers the more general problem of estimating a scale parameter in the presence of an unknown location parameter under bowl-shaped loss leading to a different class of dominating estimators. In this paper, we extend both the Stein and Brown arguments to two independent normal populations.

We use the scale invariant quadratic loss function of the form

$$L(\theta,\delta) = (\theta - \delta)^2\theta^{-2}. \tag{1.1}$$

Let $\bar{X}_i = n_i^{-1} \sum_{j=1}^{n_i} X_{ij}$, $S_i = \sum_{j=1}^{n_i}(X_{ij} - \bar{X}_i)^2$ and $T_i = n_i\bar{X}_i^2$, $i = 1,2$. Then $(\bar{X}_1,\bar{X}_2,S_1,S_2)$ is a version of the complete sufficient statistic for $(\xi_1,\xi_2,\sigma_1^2,\sigma_2^2)$.

As is well known (Stein, 1964; Brown, 1968), the best scale invariant estimator $(n_i + 1)^{-1} S_i$ is inadmissible for σ_i^2 under squared error loss. In Section 2, we pursue further the ideas of Stein and Brown. Our work there is related to that of Brewster and Zidek (1974) and Strawderman (1974). Our findings enable us to develop in Section 3 several estimators of θ which dominate

$$\delta_0 = (n_2-5)S_1/(n_1+1)S_2 , \tag{1.2}$$

the best invariant estimator of θ under loss (1.1). A brief presentation of the results of Monte Carlo simulation to measure percent improvement in risk is included.

In concluding this section, we argue that if $U_i \sim \sigma_i^2\chi^2_{n_i}$, $i = 1,2$ independent, then

$$\delta = (n_2-4)U_1/(n_1+2)U_2 \tag{1.3}$$

is admissible for $\theta = \sigma_1^2/\sigma_2^2$ under (1.1), provided $n_2 \geq 5$. This will imply that if ξ_1,ξ_2 are known by taking $U_i = \sum_{j=1}^{n_i}(X_{ij} - \xi_i)^2$, $i = 1,2$, the pair (U_1,U_2) is complete and sufficient for (σ_1^2,σ_2^2)

and δ is an admissible estimator of the variance ratio. It also implies that δ_0 in (1.2) is an admissible estimator of θ in the class of rules based upon (S_1,S_2).

The admissibility of (1.3) may be argued straightforwardly from Brown and Fox (1974, pp. 808-810). In particular, letting $W = \log(U_1/U_2)$, $V = U_2$, $\eta = \log\theta$, $\phi = \sigma_2^2$ places the joint density of W and V in their form (3), p. 809. Using the generalized prior $d\phi/\phi$ produces, by elementary calculation, (1.3) as an invariant Bayes procedure. Their regularity conditions for admissibility (a) - (d), p. 808, will be satisfied if $n_2 \geq 5$. (Condition (d), the most difficult to verify, holds by appealing to Brown (1966, Theorem 2.3.3, p. 1105).) We remark that if we use a gamma prior on ϕ, i.e., $f_{\alpha,\beta}(\phi) = \alpha^\beta\, \phi^{\beta-1}\, e^{-\alpha\phi}/\Gamma(\beta)$, the same argument will produce

$$\frac{n_2 + 2\beta - 4}{n_1 + 2}\,\frac{U_2 + \alpha}{U_2}\cdot\frac{U_1}{U_2} \tag{1.4}$$

as an admissible estimator again if $n_2 \geq 5$. Thus (1.3) is seen to be the limit of admissible Bayes as well as generalized Bayes.

2. IMPROVED ESTIMATORS OF POWERS OF VARIANCE

In this section we study the problem of estimating arbitrary powers of variance of a normal distribution under the scale invariant squared error loss. Suppose that $X_1,\dots,X_n$ is a random sample from $N(\xi,\sigma^2)$. Although $\Sigma(X_i - \bar{X})^2/(n + 1)$ is the best estimator of σ^2 in the class $c\Sigma(X_i - \bar{X})^2$ under scale invariant squared error loss, Stein (1964) showed that for any fixed ξ_0,

$$\min\left(\frac{\Sigma(X_i - \xi_0)^2}{n + 2}, \frac{\Sigma(X_i - \bar{X})^2}{n + 1}\right) \tag{2.1}$$

dominates $\Sigma(X_i - \bar{X})^2/(n+1)$ under the loss

$$L(\sigma^2,a) = (\sigma^2 - a)^2\sigma^{-4}. \tag{2.2}$$

In what follows, for convenience we take $\xi_0 = 0$ and let $S = \Sigma(X_i - \bar{X})^2$, $T = n\bar{X}^2$.

In estimating σ^{2m}, assuming $E(S^{2m})$ exists, $c_{n-1,m}S^m$ where

$c_{n-1,m} = \dfrac{\Gamma(\frac{n-1}{2} + m)2^{-m}}{\Gamma(\frac{n-1}{2} + 2m)}$ is best in the class cS^m under the loss

$$L(\sigma^{2m},a) = (\sigma^{2m} - a)^2\ \sigma^{-4m}. \tag{2.3}$$

Again, $c_{n-1,m}S^m$ is inadmissible under the loss (2.3). Following Stein's approach, we consider a class of estimators of the form $\beta(S/(S+T))(S+T)^m$ for σ^{2m} where $\beta(\cdot)$ is a function from $[0,1] \to [0,\infty)$. Using loss (2.3), the risk of such estimators depends only upon ξ^2/σ^2 (so that without loss of generality, we set $\sigma^2 = 1$) and may be written (e.g., Strawderman (1974), p. 191)

$$1 + E\left[\left(\beta\left(\frac{S}{S+T}\right) - c_{n+2L,m}\right)^2 \frac{\Gamma(\frac{n}{2}+L+2m)2^{2m}}{\Gamma(\frac{n}{2}+L)} - \left(\frac{\Gamma(\frac{n}{2}+L+m)}{\Gamma(\frac{n}{2}+L+2m)2^m}\right)^2\right] \tag{2.4}$$

where the expectation is over the joint distribution of $S/(S+T)$ and L, with $S/(S+T)$ having a beta distribution $Be(\frac{n-1}{2}, \frac{2L+1}{2})$ and L having a Poisson distribution with parameter $n\xi^2/2\sigma^2$. Clearly, $c_{n+2L,m}$ increases in L for $m < 0$ and decreases in L for $m > 0$. Hence, if $\beta(S/(S+T))(S+T)^m = c_{n-1,m}S^m$, i.e., $\beta = c_{n-1,m}(\frac{S}{S+T})^m$, then one can choose for $m > 0$, $\beta^* = \min(\beta, c_{n,m})$ and for $m < 0$, $\beta^* = \max(\beta, c_{n,m})$. From (2.4) we see that $\beta^*(S/(S+T))(S+T)^m$ dominates $c_{n-1,m}S^m$ under loss (2.3). We state this result as

Theorem 2.1. For $X_1,\ldots,X_n$ a random sample from $N(\xi,\sigma^2)$, in estimating σ^m, $m > -(n-1)/4$ (to insure $c_{n-1,m}$ finite) under scale invariant loss (2.3) if

$$\begin{aligned} &m > 0,\ \min(c_{n,m}(S+T)^m, c_{n-1,m}S^m) \text{ dominates } c_{n-1,m}S^m \\ &m < 0,\ \max(c_{n,m}(S+T)^m, c_{n-1,m}S^m) \text{ dominates } c_{n-1,m}S^m. \end{aligned} \tag{2.5}$$

In particular, when $m = -1$, if $n \geq 5$, $\max(\frac{n-4}{S+T}, \frac{n-5}{S})$ dominates $(n-5)/S$.

Brown (1968) has created a different type of estimator for σ^2 (i.e., $m = 1$) to dominate $S/(n+1)$. In the case of normality, his estimator takes the form

$$\begin{array}{lll} S/(n+1) & \text{if} & T/S > c \\ a^*S & \text{if} & T/S < c \end{array} \tag{2.6}$$

where $a^* < 1/(n+1)$ and depends on c. In what follows, we give a convenient expression for a^* and relate the Brown and Stein approaches. Brewster and Zidek (1974, p. 22) look more generally into the relationship between the Stein and Brown approaches in the context of dominance of equivariant estimators.

Using (2.4) at $m = 1$, any estimator of the form

$$\delta(S,T) = \beta(S/(S+T))(S+T) \tag{2.7}$$

under the invariant loss (2.3) has risk of the form

$$R(\sigma^2,\delta(S,T)) = E[\beta(\frac{S}{S+T}) - (n+2L+2)^{-1}]^2(n+2L)(n+2L+2) + \rho(\xi^2/\sigma^2), \tag{2.8}$$

where $\rho = E2(n + 2L + 2)^{-1}$.

By noting that

$$T/S \gtrless c \leftrightarrow \frac{S}{S+T} \lessgtr d = (1 + c)^{-1}$$

and defining

$$\beta_{d,a^*}(y) = (n + 1)^{-1}I_{(0,d)}(y)y + a^*I_{(d,\infty)}(y)y$$

where I is the indicator function, the Brown estimator is

$$\beta_{d,a^*}(\frac{S}{S+T})(S+T) , \tag{2.9}$$

i.e., of the form (2.7) with risk as in (2.8).

The best choice of a^* in (2.9) may be obtained from expression (6.2) of Brown (1968). After some manipulation, we may write it explicitly in the form

$$a^* = (n+1)^{-1} I_{1-d}(1/2, \frac{n+1}{2})/I_{1-d}(1/2, \frac{n+3}{2}) \qquad (2.10)$$

where $I_x(a,b)$ is the incomplete beta function (see, e.g., Abramowitz and Stegun, 1965). In fact, for our case, we can simplify the argument in Theorem 4.1 of Brown by considering (2.8) in the space of $(S/(S+T),L)$. We omit the details.

We note that the Stein improved estimator (with $\xi_0 = 0$) may be written in a form similar to (2.6), i.e.,

$$\begin{array}{ll} S/(n+1) & \text{if } T/S > (n+1)^{-1} \\ (S+T)/(n+2) & \text{if } T/S < (n+1)^{-1} \end{array} \qquad (2.11)$$

In fact, if we use $c = (n+1)^{-1}$ in (2.6), then $d = (n+1)/(n+2)$. Looking at (2.10) we see that if d is much smaller than $(n+1)/(n+2)$ (so that c is larger than $(n+1)^{-1}$), a^* is essentially $(n+1)^{-1}$ and (2.6) is essentially $S/(n+1)$. But also if d is larger than $(n+1)/(n+2)$, i.e., nearly 1, then c is very small, so that we will take a^*S with very small probability and again (2.6) is essentially $S/(n+1)$. Hence, c in the vicinity of $(n+1)^{-1}$ seems best and our numerical studies support this.

Brown (1968) discusses improved estimation of σ^m for $m > 0$. In fact, analogous to Theorem 2.1, we can show

Theorem 2.2. Suppose $X_1,\ldots,X_n$ is a random sample from $N(\xi,\sigma^2)$, in estimating σ^m, $m > -(n-1)/4$ under loss (2.3)

$$\begin{array}{ll} c_{n-1,m}S^m & \text{if } T/S > c \\ a_m^* S^m & \text{if } T/s < c \end{array} \qquad (2.12)$$

dominates $c_{n-1,m}S^m$ in terms of risk, where

$$a_m^* = c_{n-1,m}\, I_{1-d}(1/2, \tfrac{n+2m-1}{2})/I_{1-d}(1/2, \tfrac{n+4m-1}{2}) \qquad (2.13)$$

Proof. The proof is essentially that of Brown's Theorem 4.1. When $m < 0$ the inequality analogous to his expression (4.18) is reversed. Notationally we have suppressed the dependence of a_m^* upon n and d.

Remark 2.1. If $m > 0$, $a_m^* < c_{n-1,m}$, if $m < 0$, $a_m^* > c_{n-1,m}$. This is analogous to using a minimum or maximum according to $m > 0$ or $m < 0$ as in (2.5). Alternatively, the estimators in (2.5) may be written in the same form as (2.12), i.e.,

$$\begin{array}{ll} c_{n-1,m}S^m & \text{if } T/S > c_{n-1,m} \\ c_{n,m}(S+T)^m & \text{if } T/S < c_{n-1,m}\,. \end{array}$$

Therefore, the discussion following (2.11) suggests taking c in (2.12) in the vicinity of $c_{n-1,m}$.

Remark 2.2. Brown (1968) has considered more generally the estimator of σ^m, $m > 0$, in distributional families when σ is a scale parameter in the presence of an unknown location parameter under bowl-shaped loss.

3. IMPROVED ESTIMATORS OF THE VARIANCE RATIO

In this section we will show that when the means are unknown, the best invariant estimator of the variance ratio $\theta = \sigma_1^2/\sigma_2^2$ is inadmissible. We will obtain several improved estimators of θ.

From the notation developed in Section 1, it follows that the best invariant estimator of θ (when the means ξ_1 and ξ_2 are unknown) is

$$\delta_0 = \frac{(n_2-5)}{n_1+1}\,\frac{S_1}{S_2}\,. \qquad (3.1)$$

Theorems 3.1 and 3.2 show that the obvious estimators created either by improving upon the best invariant estimator of σ_1^2 or of σ_2^{-2} dominates (3.1).

Theorem 3.1. Under the loss (1.1), δ_0 is inadmissible for θ and is dominated by

$$\delta_1^S = \min(\delta_0, \frac{n_2-5}{n_1+2} \quad \frac{S_1+T_1}{S_2}) \tag{3.2}$$

and

$$\delta_2^S = \max(\delta_0, \frac{n_2-4}{n_1+1} \quad \frac{S_1}{S_2+T_2}). \tag{3.3}$$

Proof. The proof is essentially that of Theorem 2.1 using, for example, in the first case the fact that S_2 follows up to a constant a central chi-square distribution independent of $S_1 + T_1$.

Remark 3.1. Theorem 3.1 could clearly be extended to provide dominating estimators for $\sigma_1^{2m_1}/\sigma_2^{2m_2}$. We omit the details here.

Taking $m_1 = m_2 = 1$, we note that for estimators of the form $\beta_1(S_1/(S_1+T_1))\ \beta_2(S_2/(S_2+T_2))(S_1+T_1)/(S_2+T_2)$ the risk depends only upon ξ_1^2/σ_1^2 and ξ_2^2/σ_2^2. So without loss of generality, we may take $\sigma_i^2 = 1$ and hence $\theta = 1$. Observing that

$$\delta_1^S = \frac{n_2-5}{S_2} \min((S_1+T_1)/(n_1+2), S_1/(n_1+1))$$

we see that Stein's result is

$$E\{\min(\frac{S_1+T_1}{n_1+2}, \frac{S_1}{n_1+1}) - 1\}^2 \leq E(\frac{S_1}{n_1+1} - 1)^2 \quad \forall \xi_1$$

while we have added

$$E\{\min(\frac{S_1+T_1}{n_1+2}, \frac{S_1}{n_1+1}) \frac{n_2-5}{S_2} - 1\}^2 \leq E(\frac{S_1}{n_1+1} \frac{n_2-5}{S_2} - 1)^2 \quad \forall \xi_1.$$

Thus we have the following remark.

Remark 3.2. More generally using the same argument as in Theorem 3.1 and assuming the expectations exist.

$$E\{\beta^*(\frac{S_1}{S_1+T_1})(S_1+T_1) - 1\}^2 \leq E\{\beta(\frac{S_1}{S_1+T_1})(S_1+T_1) - 1\}^2 \quad \forall \xi_1$$

if and only if

$$E\{\beta^*(\frac{S_1}{S_1+T_1})(S_1+T_1)h(S_2) - 1\}^2 \leq E\{\beta(\frac{S_1}{S_1+T_1})(S_1+T_1)h(S_2) - 1\}^2 \quad \forall \xi_1 .$$

Similarly

$$E\{\beta^*(\frac{S_2}{S_2+T_2})(S_2+T_2)^{-1} - 1\}^2 \leq E\{\beta(\frac{S_2}{S_2+T_2})(S_2+T_2)^{-1} - 1\}^2 \quad \forall\xi_2$$

if and only if

$$E\{\beta^*(\frac{S_2}{S_2+T_2})\frac{h(S_1)(S_2+T_2)^{-1}}{n_1+1} - 1\}^2 \leq E\{\beta(\frac{S_2}{S_2+T_2})\frac{h(S_1)(S_2+T_2)^{-1}}{n_1+1} - 1\}^2 \quad \forall\xi_2.$$

Remark 3.3. It is noteworthy that the argument for Theorem 3.1 and Remark 3.2 depends entirely upon using S_2 with estimates of the form $\beta_1(\frac{S_1}{S_1+T_1})(S_1+T_1)$ (or vice versa) rather than using $S_2 + T_2$.

Thus we cannot show, for instance, that the appealing estimator

$$\delta_6^S = \min(\frac{S_1+T_1}{n_1+2}, \frac{S_1}{n_1+1}) \max(\frac{n_2-4}{S_2+T_2}, \frac{n_2-5}{S_2}) \tag{3.4}$$

dominates δ_1^S, δ_2^S or even δ_0. We will return to this point later.

With regard to the Brown estimators, we have

Theorem 3.2. *Under the loss (1.1), δ_0 is inadmissible for θ and is* dominated by

$$\delta_1^B = \begin{cases} \delta_0 & \text{if } T_1/S_1 > c \\ a^*S_1 \cdot \frac{n_2-5}{S_2} & \text{if } T_1/S_1 < c \end{cases} \tag{3.5}$$

and

$$\delta_2^B = \begin{cases} \delta_0 & \text{if } T_2/S_2 > c \\ \frac{a^{**}}{n_1+1} (\frac{S_1}{S_2}) & \text{if } T_2/S_2 < c \end{cases} \tag{3.6}$$

where a^* is given in (2.10) and $a^{**} \equiv a^*_{-1}$ in (2.13).

Proof. Consider δ_1^B. We may write

$$\delta_1^B = \beta_{d,a^*}(\frac{S_1}{S_1+T_1})(S_1+T_1)\frac{n_2-5}{S_2}$$

and appeal to Remark 3.2. The argument is similar for δ_2^B.

Remark 3.4. Extension of Theorem 3.2 in the direction of Remarks 2.2 and 3.1 is straightforward. We omit the details.

The estimators δ_i^S and δ_i^B, $i = 1,2$, are not admissible. We next provide explicit estimators which dominate them. In the process we clarify the problem of estimation of θ. Suppose that

$$\alpha_1 = S_1/(n_1+1),\ \alpha_2^S = (S_1+T_1)/(n_1+2),\ \alpha_2^B = a^*S_1$$

$$\beta_1 = (n_2-5)/S_2,\ \beta_2^S = (n_2-4)/(S_2+T_2),\ \beta_2^B = a^{**}/S_2$$

and define regions

$$\begin{aligned} A_1 &= \{T_1/S_1 > (n_1+1)^{-1},\ T_2/S_2 > (n_2-5)^{-1}\} \\ A_2 &= \{T_1/S_1 < (n_1+1)^{-1},\ T_2/S_2 > (n_2-5)^{-1}\} \\ A_3 &= \{T_1/S_1 > (n_1+1)^{-1},\ T_2/S_2 < (n_2-5)^{-1}\} \\ A_4 &= \{T_1/S_1 < (n_1+1)^{-1},\ T_2/S_2 < (n_2-5)^{-1}\}. \end{aligned} \tag{3.7}$$

Then Figure 1 shows the previously discussed δ's and several additional ones. In looking at the dominance of δ_1 over δ_0, we can show that is always accomplished on A_2, whereas the situation is unclear on A_4. Similarly the dominance of δ_2 over δ_0 is always accomplished on A_3, again with the situation on A_4 unclear. The following theorem gives the dominance results mentioned after Remark 3.4.

<u>Theorem 3.3</u>. Under the loss (1.1) the following hold:

$$\delta_3^S \text{ dominates } \delta_1^S,\ \delta_3^B \text{ dominates } \delta_1^B,$$

$$\delta_4^S \text{ dominates } \delta_2^S \text{ and } \delta_4^B \text{ dominates } \delta_2^B.$$

<u>Proof</u>. Immediate from the Lemma 1 of the appendix and the definition of the δ's.

<u>Corollary 3.1</u>. δ_5^S dominates δ_0 and δ_5^B dominates δ_0.

<u>Remark 3.5</u>. For the Brown estimators, one can use general c_1, c_2 in defining the A_i's and the above results will go through.

<u>Remark 3.6</u>. The argument in Lemma 1 of the appendix fails on A_4 and, in fact, there is no best choice on A_4. For example, the intuitively appealing estimator δ_6^S in Figure 1 suggests $\alpha_2^S\beta_2^S$ on A_4,

Estimators	A_1	A_2	A_3	A_4
δ_0	$\alpha_1\beta_1$	$\alpha_1\beta_1$	$\alpha_1\beta_1$	$\alpha_1\beta_1$
δ_1^S	$\alpha_1\beta_1$	$\alpha_2^S\beta_1$	$\alpha_1\beta_1$	$\alpha_2^S\beta_1$
δ_1^B	$\alpha_1\beta_1$	$\alpha_2^B\beta_1$	$\alpha_1\beta_1$	$\alpha_2^B\beta_1$
δ_2^S	$\alpha_1\beta_1$	$\alpha_1\beta_1$	$\alpha_1\beta_2^S$	$\alpha_1\beta_2^S$
δ_2^B	$\alpha_1\beta_1$	$\alpha_1\beta_1$	$\alpha_1\beta_2^B$	$\alpha_1\beta_2^B$
δ_3^S	$\alpha_1\beta_1$	$\alpha_2^S\beta_1$	$\alpha_1\beta_2^S$	$\alpha_2^S\beta_1$
δ_3^B	$\alpha_1\beta_1$	$\alpha_2^B\beta_1$	$\alpha_1\beta_2^B$	$\alpha_2^B\beta_1$
δ_4^S	$\alpha_1\beta_1$	$\alpha_2^S\beta_1$	$\alpha_1\beta_2^S$	$\alpha_1\beta_1^S$
δ_4^B	$\alpha_1\beta_1$	$\alpha_2^B\beta_1$	$\alpha_1\beta_2^B$	$\alpha_1\beta_2^B$
δ_5^S	$\alpha_1\beta_1$	$\alpha_2^S\beta_1$	$\alpha_1\beta_2^S$	$\alpha_1\beta_1$
δ_5^B	$\alpha_1\beta_1$	$\alpha_2^B\beta_1$	$\alpha_1\beta_2^B$	$\alpha_1\beta_1$
δ_6^S	$\alpha_1\beta_1$	$\alpha_2^S\beta_1$	$\alpha_1\beta_2^S$	$\alpha_2^S\beta_2^S$
δ_6^B	$\alpha_1\beta_1$	$\alpha_2^B\beta_1$	$\alpha_1\beta_2^B$	$\alpha_2^B\beta_2^B$

FIG. 1. Estimators of θ Defined by Regions

but we cannot show that δ_6^S dominates even δ_0. However on A_4 (suppressing superscripts), $\alpha_1 > \alpha_2$, $\beta_1 < \beta_2$ implies $\alpha_2\beta_1 < \alpha_1\beta_1 < \alpha_1\beta_2$ and $\alpha_2\beta_1 < \alpha_2\beta_2 < \alpha_1\beta_2$. This suggests that $\alpha_1\beta_1$ will be close to $\alpha_2\beta_2$; that is, the performance of δ_6 will be essentially that of δ_5 which does dominate δ_0. Our numerical studies show that δ_5 or δ_6 is nearly always best supporting a middle rather than an extreme choice on A_4.

Monte Carlo simulations were performed to obtain risks for the estimators in Figure 1. 10,000 replications were used. The percentage improvements in risk with respect to δ_0 for δ_3^S, δ_4^S, δ_5^S, δ_6^S, δ_3^B, δ_4^B, δ_5^B, δ_6^B are given in Table I for the case $n_1 = n_2 = 10$, $\sigma_1^2 = \sigma_2^2 = 1$ and a range of ξ_1^2, ξ_2^2. As expected the performance of δ_6 is close to that of δ_5 although generally a bit better. The

percent improvements are small and become smaller with increasing n_i. However they are greater than those observed by Brown (1968). His brief numerical study showed for a single variance under squared error loss a maximum improvement of 1 to 2%. For a variance ratio we are able to roughly double this. Moreover, the simplicity of the Stein estimators encourages their use particularly for small n_1, n_2.

TABLE I

Percentage Improvements in Risks* over δ_0

$n_1 = n_2 = 10$

(ξ_1^2, ξ_2^2)	δ_3^S	δ_4^S	δ_5^S	δ_6^S	δ_3^B	δ_4^B	δ_5^B	δ_6^B
(0,0)	2.0	4.0	2.9	4.1	2.0	3.5	3.0	3.5
(0,.01)	2.1	4.0	3.1	4.0	2.0	3.8	3.0	3.7
(0,.1)	1.5	3.2	2.5	2.9	1.7	3.5	2.6	3.3
(0,1)	0.3	0.6	0.4	0.4	0.5	0.8	0.6	0.7
(.01,0)	2.5	4.0	3.2	4.2	2.4	3.4	3.2	3.6
(.01,.01)	2.2	4.1	3.1	4.0	2.0	3.9	3.0	3.7
(.01,.1)	1.6	3.4	2.4	3.1	1.8	3.5	2.6	3.3
(.01,1)	0.4	0.7	0.6	0.6	0.5	0.9	0.6	0.7
(.1,0)	3.5	4.1	3.9	4.3	2.9	3.3	3.4	3.5
(.1,.01)	3.3	4.0	3.8	4.2	2.8	3.6	3.4	3.7
(.1,.1)	2.6	3.0	3.2	3.3	2.7	3.3	3.2	3.4
(.1,1)	0.8	1.0	0.9	0.9	0.8	1.0	0.9	0.9
(1,0)	3.1	3.0	3.1	3.0	2.4	2.3	2.3	2.3
(1,.01)	2.9	2.8	2.8	2.9	2.7	2.5	2.6	2.6
(1,.1)	2.2	2.2	2.2	2.2	2.4	2.4	2.4	2.4
(1,1)	0.3	0.3	0.3	0.3	0.4	0.4	0.4	0.4

*The largest sample standard error over all the 128 percentage improvements was less than .2.

APPENDIX

Lemma 1. Under the notations in (3.7) with α_2 either α_2^S or α_2^B and β_2 either β_2^S or β_2^B, the following inequalities hold:

$$\text{(i)}\quad EI_{A_2 \cup A_4}(\alpha_2\beta_1 - 1)^2 \leq EI_{A_2 \cup A_4}(\alpha_1\beta_1 - 1)^2 \quad \forall \xi_1$$

$$\Rightarrow \quad EI_{A_2}(\alpha_2\beta_1 - 1)^2 \leq EI_{A_2}(\alpha_1\beta_1 - 1)^2 \quad \forall \xi_1 \qquad \text{(A.1)}$$

$$\text{(ii)}\quad EI_{A_3 \cup A_4}(\alpha_1\beta_2 - 1)^2 \leq EI_{A_3 \cup A_4}(\alpha_1\beta_1 - 1)^2 \quad \forall \xi_2$$

$$\Rightarrow \quad EI_{A_3}(\alpha_1\beta_2 - 1)^2 \leq EI_{A_3}(\alpha_1\beta_1 - 1)^2 \quad \forall \xi_2 \qquad \text{(A.2)}$$

Proof. Simple calculation shows that (A.1) is equivalent to

$$EI_{A_2 \cup A_4} \beta_1^2 (\alpha_1^2 - \alpha_2^2) \geq 2EI_{A_2 \cup A_4} \beta_1 (\alpha_1 - \alpha_2). \tag{A.3}$$

Note that on $A_2 \cup A_4$, $\alpha_1 > \alpha_2$. Thus (A.3) becomes

$$\frac{\int_{\alpha_1 > \alpha_2} (\alpha_1 - \alpha_2) d\alpha_1 d\alpha_2}{\int_{\alpha_1 > \alpha_2} (\alpha_1^2 - \alpha_2^2) d\alpha_1 d\alpha_2} \leq E(\beta_1^2)/2E(\beta_1).$$

We need only show that

$$\frac{E(\beta_1^2)}{E(\beta_1)} \leq \frac{E(\beta_1^2 | T_2/S_2 > (n_2-5)^{-1})}{E(\beta_1 | T_2/S_2 > (n_2-5)^{-1})}. \tag{A.4}$$

Define the density $p(s_2)$ of S_2 by

$$p(s_2) = \frac{\beta_1(s_2) \chi^2_{n_2-1}(s_2)}{\int \beta_1(s_2) \chi^2_{n_2-1}(s_2) ds_2}.$$

Then for each fixed t_2, taking expectations w.r.t. the pdf $p(s_2)$, one gets

$$E_p(\beta_1) E_p[I_{(0,(n_2-5)t_2)}] \leq E_p[\beta_1 \; I_{(0,(n_2-5)t_2)}]. \tag{A.5}$$

Taking expectations over t_2 of both sides of (A.5) and simple manipulation yields (A.4). The proof of (A.2) is similar.

ACKNOWLEDGMENT

The authors acknowledge John Judge and Brad Carlin for performing the computations.

The research was supported in part by the University of Connecticut Research Foundation.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 403	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): On The Estimation Of A Variance Ratio	5. TYPE OF REPORT & PERIOD COVERED: TECHNICAL REPORT
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Alan E. Gelfand and Dipak K. Dey	8. CONTRACT OR GRANT NUMBER(s): N00014-86-K-0156
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Department of Statistics, Stanford University, Stanford, CA 94305	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: NR-042-267
11. CONTROLLING OFFICE NAME AND ADDRESS: Office of Naval Research, Statistics & Probability Program Code 1111	12. REPORT DATE: April 6, 1988
	13. NUMBER OF PAGES: 16
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

Partial support from University of Connecticut Research Foundation.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Variance ratio; loss function; invariance; admissibility; inadmissibility.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The estimation of the ratio of two independent normal variances is considered under scale invariant squared error loss function, when the means are unknown. The best invariant estimator is shown to be inadmissible. Two new classes of improved estimators are obtained, one by extending Stein (1964) and the other by extending Brown (1968). Numerical studies are presented to indicate the percent improvements in risk.

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